7.6 Complex-Valued Eigenvalues

We have our system of n first order homogeneous equations

$$\vec{x}' = A\vec{x}$$

Solutions are of the form

$$\vec{x} = \vec{\xi} e^{rt}$$

where

$$\det(A - rI) = 0$$

and

$$(A - rI)\vec{\xi} = 0 .$$

We have examined the case when we have real eigenvalues.

A is a real matrix, so complex roots occur in pairs $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$

where λ and μ are real. For the corresponding eigenvectors, $\vec{\xi}_1$ is the eigenvector for r_1. Then

$$(A - r_1 I)\vec{\xi}_1 = 0$$

$$\overline{(A - r_1 I)\vec{\xi}_1} = \bar{0}$$

$$(A - \bar{r}_1 I)\overline{\vec{\xi}_1} = 0$$

$$(A - r_2 I)\overline{\vec{\xi}_1} = 0$$

So $\vec{\xi}_2 = \overline{\vec{\xi}_1}$.

Therefore, our solutions are of the form

$$\vec{x}^{(1)} = \vec{\xi}^{(1)} e^{r_1 t} \quad \text{and} \quad \vec{x}^{(2)} = \overline{\vec{\xi}^{(1)}} e^{\bar{r}_1 t}$$

Now, we may write $\vec{\xi}^{(1)} = \vec{a} + i\vec{b}$ where

$\vec{a}$ and $\vec{b}$ are real, so

$$
\begin{aligned}
\vec{x}^{(1)}(t) &= \vec{\xi}^{(1)} e^{r_1 t} \\
&= (\vec{a} + i\vec{b}) e^{(\lambda + i\mu)t} \\
&= (\vec{a} + i\vec{b}) e^{\lambda t}(\cos(\mu t) + i\sin(\mu t)) \\
&= (\vec{a}\cos(\mu t) - \vec{b}\sin(\mu t)) e^{\lambda t} \\
&\quad + i(\vec{a}\sin(\mu t) + \vec{b}\cos(\mu t)) e^{\lambda t}
\end{aligned}
$$

so

$$\vec{u} = (\vec{a}\cos(\mu t) - \vec{b}\sin(\mu t)) e^{\lambda t}$$

$$\vec{v} = (\vec{a}\sin(\mu t) + \vec{b}\cos(\mu t)) e^{\lambda t}$$

are our real-valued solutions.

Ex $\quad \vec{x}\,' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \vec{x}$

The eigenvalues are

$$\det\begin{pmatrix} -\frac{1}{2}-\lambda & 1 \\ -1 & -\frac{1}{2}-\lambda \end{pmatrix} = 0$$

$$\left(-\frac{1}{2}-\lambda\right)^2 + 1 = 0$$

$$\lambda^2 + \lambda + \frac{5}{4} = 0$$

so $$\lambda = \frac{-1 \pm \sqrt{1-5}}{2} = -\frac{1}{2} \pm i$$

Next, the eigenvectors are

$\underline{r_1 = -\frac{1}{2}+i}$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \vec{\xi}^{(1)} = 0$$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$$

$$-i\xi_1 + \xi_2 = 0$$

$$\xi_2 = i\xi_1$$

$$\vec{\xi}^{(1)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ i\xi_1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \xi_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\xi_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Therefore,

$$\vec{u} = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t \right) e^{-t/2}$$

$$\vec{v} = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t \right) e^{-t/2}$$

Thus,

$$\vec{x} = c_1 \vec{u} + c_2 \vec{v}$$

$$= c_1 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t \right) e^{-t/2}$$

$$+ c_2 \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t \right) e^{-t/2}$$

$$= \begin{pmatrix} c_1 \cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{pmatrix} e^{-t/2}$$

Ex $\vec{x}' = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} \vec{x}$

The eigenvalues are

$$\det \begin{pmatrix} \alpha - \lambda & 2 \\ -2 & -\lambda \end{pmatrix} = 0$$

$$(\alpha - \lambda)(-\lambda) + 4 = 0$$

$$\lambda^2 - \alpha\lambda + 4 = 0$$

$$\lambda_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}$$

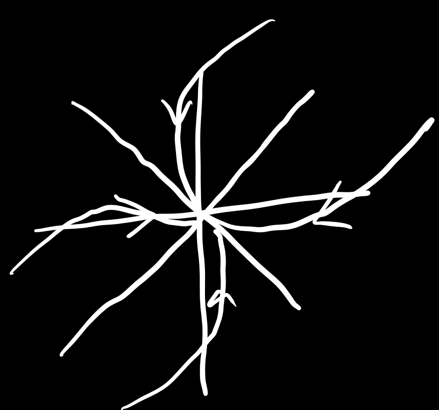
Let $\vec{\xi}^{(1)} = \vec{a} + i\vec{b}$ be the eigenvector corresponding to λ_1.

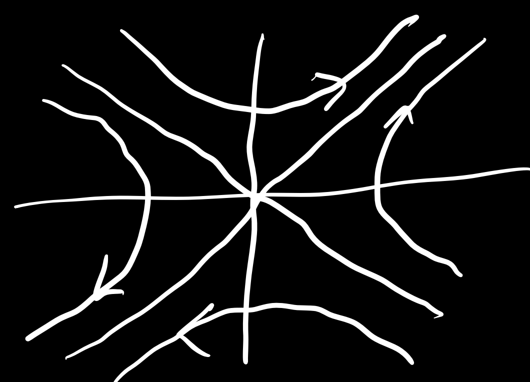
What do solutions look like?

$\underline{\alpha > 0 \text{ and } \alpha^2 > 16}$

$$\vec{x} = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

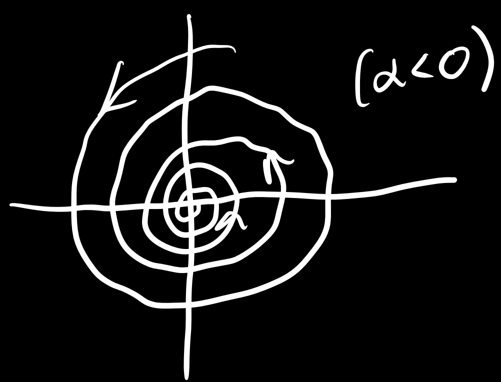
Nodes $r_1, r_2 < 0$ Saddles, $r_1 > 0 > r_2$



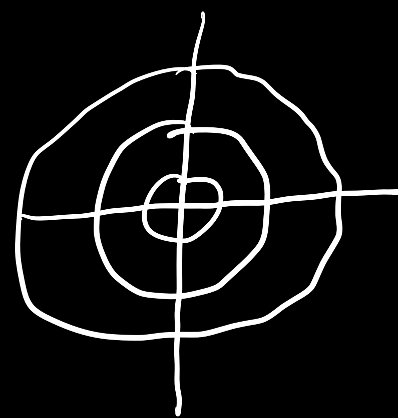


$\underline{\alpha^2 < 16}$ $\Rightarrow$ complex roots

Spiral



Center $(\alpha = 0)$



7.7 Fundamental Matrices

Suppose that $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$ form a fundamental set of solutions for

$$\vec{x}\,' = P(t)\vec{x}.$$

The matrix

$$\Psi = \begin{pmatrix} | & | & & | \\ \vec{x}^{(1)} & \vec{x}^{(2)} & \cdots & \vec{x}^{(n)} \\ | & | & & | \end{pmatrix}$$

is called the fundamental matrix for

the system.

The general solution is

$$\vec{x}(t) = c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} + \cdots + c_n \vec{x}^{(n)}$$

or

$$\vec{x}(t) = \Psi(t) \vec{c} \quad , \text{ where } \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

The IC $\vec{x}(t_0) = \vec{x}^0$, so

$$\Psi(t_0) \vec{c} = \vec{x}(t_0)$$

$$\Psi(t_0) \vec{c} = \vec{x}^0$$

$$\vec{c} = \Psi^{-1}(t_0) \vec{x}^0$$

Therefore,

$$\vec{x}(t) = \Psi(t) \Psi^{-1}(t_0) \vec{x}^0 \quad .$$

The Matrix Exponential

Recall
$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{(at)^n}{n!}$$

and the scalar IVP

$$x' = ax, \quad x(0) = x_0$$

has solution
$$x = x_0 e^{at}$$

Can we find something similar for
$$\vec{x}\,' = A\vec{x}, \quad \vec{x}(0) = \vec{x}^0 \,?$$

Let's plug the matrix A into the Taylor series,

$$e^{At} = I + \sum_{n=1}^{\infty} \frac{(At)^n}{n!}$$

So

$$\frac{d}{dt} e^{At} = \frac{d}{dt}\left(I + \sum_{n=1}^{\infty} \frac{(At)^n}{n!}\right)$$

$$= \sum_{n=1}^{\infty} \frac{n(At)^{n-1}}{n!} A$$

$$= \sum_{n=1}^{\infty} \frac{(At)^{n-1}}{(n-1)!} A$$

$$= A + \sum_{n=2}^{\infty} \frac{(At)^{n-1}}{(n-1)!} A$$

$$= A + \sum_{n=1}^{\infty} \frac{(At)^n}{n!} A$$

$$= A\left(I + \sum_{n=1}^{\infty} \frac{(At)^n}{n!}\right)$$

$$= A e^{At}$$

So $\vec{x} = e^{At}$

Satisfies the system of DEs.

Using the IC, $\vec{x}(0) = \vec{x}^0$, we get

$$\vec{x} = e^{At}\,\vec{x}^0 .$$

Diagonalizable Matrices

Recall If a matrix A is diagonalizable, we may write

$$A = T D T^{-1}$$

where D is a diagonal matrix whose entries are the eigenvalues of A

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

and

$$T = \begin{pmatrix} | & | & & | \\ \vec{\xi}^{(1)} & \vec{\xi}^{(2)} & \cdots & \vec{\xi}^{(n)} \\ | & | & & | \end{pmatrix}$$

where $\vec{\xi}^{(1)}, \cdots, \vec{\xi}^{(n)}$ are the corresponding eigenvectors.

Now, the matrix exponential becomes

$$e^{At} = I + \sum_{n=1}^{\infty} \frac{(At)^n}{n!}$$

$$= I + \sum_{n=1}^{\infty} \frac{(TDT^{-1})^n t^n}{n!}$$

$$(TDT^{-1})^n = TD\underbrace{T^{-1}T}D\underbrace{T^{-1}\cdots\cdots T}DT^{-1}$$

$$= TD^nT^{-1}$$

$$= I + \sum_{n=1}^{\infty} \frac{TD^nT^{-1}t^n}{n!}$$

$$= T\left(I + \sum_{n=1}^{\infty} \frac{D^n t^n}{n!}\right)T^{-1}$$

$$= T e^{Dt} T^{-1}$$

so

$$\vec{x} = T e^{Dt} T^{-1} \vec{x}^0 .$$

Remark

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \qquad D^n = \begin{pmatrix} \lambda_1^n & & 0 \\ & \lambda_2^n & \\ 0 & & \ddots & \\ & & & \lambda_n^n \end{pmatrix}$$

$$e^{Dt} = I + \sum_{n=1}^{\infty} \frac{D^n t^n}{n!}$$

$$= I + \sum_{n=1}^{\infty} \begin{pmatrix} \frac{\lambda_1^n t^n}{n!} & & 0 \\ & \ddots & \\ 0 & & \frac{\lambda_n^n t^n}{n!} \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \sum_{n=1}^{\infty} \frac{\lambda_1^n t^n}{n!} & & \\ & \ddots & \\ & & 1 + \sum_{n=1}^{\infty} \frac{\lambda_n^n t^n}{n!} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}$$

Ex Compute e^{At} where $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

The eigenvalues are

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)^2 - 4 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

$\lambda_1 = 3$ and $\lambda_2 = -1$.

Next, the eigenvectors are

$\underline{\lambda_1 = 3}$

$$(A - \lambda_1 I)\vec{v}_1 = 0$$

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = 0$$

$$-2v_{11} + v_{12} = 0$$

$$v_{12} = 2v_{11}$$

$$\vec{v}_1 = \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} v_{11} \\ 2v_{11} \end{pmatrix} = v_{11}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\underline{\lambda_2 = -1}$

$$(A - \lambda_2 I)\vec{v}_2 = 0$$

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = 0$$

$$2v_{21} + v_{22} = 0$$

$$v_{22} = -2v_{21}$$

$$\vec{v}_2 = \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} v_{21} \\ -2v_{21} \end{pmatrix} = v_{21} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

so

$$D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$$

and

$$T^{-1} = \frac{1}{-2-2} \begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{-1}{4} \begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{-1}{4} \end{pmatrix}$$

Therefore,

$$e^{At} = T e^{Dt} T^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{-1}{4} \end{pmatrix}$$